Gevrey Regularity Of Global Attractor For Generalized Benjamin-Bona-Mahony Equation

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Abstract

We prove the Gevrey regularity of the global attractor of the dynamical system generated by the generalized Benjamin-Bona-Mahony equation with periodic boundary conditions. This result means that elements of the attractor are real analytic functions in spatial variables. As an application we prove the existence of two determining nodes for the problems in one spatial dimension.

Key words and phrases: Benjamin-Bona-Mahony equation, global attractor, Gevrey regularity, determining nodes.

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Introduction

In the domain $\mathcal{O} = (0, L) \times (0, L) \times \cdots \times (0, L) \subset \mathbb{R}^n$, where $L > 0$, we consider the following initial-boundary value problem for the generalized Benjamin-Bona-Mahony equation:

$$u_t - a \triangle u_t - b \triangle u + \operatorname{div}\{F(u)\} = h(x), \qquad x \in \mathcal{O},\ t \in \mathbb{R}^+ \tag{0.1}$$

$$u(0, x) = u_0(x), \qquad x \in \mathcal{O}, \tag{0.2}$$

with the periodic boundary conditions:

$$u(t, x + Le_i) = u(t, x), \quad u_{x_i}(t, x + Le_i) = u_{x_i}(t, x) \tag{0.3}$$

for all $x \in \Gamma_i := \partial\mathcal{O} \cap \{x_i = 0\}$, $t \in \mathbb{R}^+$ and $i = 1, 2, \ldots, n$. Here $\{e_k\}$ is the standard basis in $\mathbb{R}^n$. We assume that a and b are positive constants, $u_0(x)$ and $h(x)$ are given functions and

$$F(u) = (F_1(u), F_2(u), \ldots, F_n(u))$$

is a given polynomial vector field, i.e.

$$F_j(u) = \sum_{k=1}^{l} b_k^j u^k, \qquad j = 1, \ldots, n, \tag{0.4}$$

where $1 \leq l < \infty$ if $n \leq 2$, $l = 1$ or $l = 2$ if $n = 3$, and $l = 1$ if $n \geq 4$. We also use the notation

$$\operatorname{div}\{F(u)\} = \sum_{i=1}^{n} F_i'(u) \cdot u_{x_i}.$$

Equation (0.1) for $n = 1$ has been proposed by Benjamin, Bona and Mahoni [5] as a model for the propagation of long waves. The model incorporates nonlinear dispersive and dissipative effects. This equation and also related types of the Benjamin-Bona-Mahony equation were studied by many authors. Results on the existence and uniqueness of solutions can be found e.g. in [3, 8, 15, 21, 26]. The long-term behavior of solutions, such as stability or the rate of decay were studied in [1, 2, 6, 19, 22, 23]. In the one spatial dimension case global attractors for problem (0.1)-(0.3) were investigated in [28]-[30]. For n spatial dimensions the existence and finite-dimensionality of the global attractor were proved in [7].

In this paper we study properties of asymptotic regularity of solutions to problem (0.1)–(0.3). Our main result (see Theorem 1.3) asserts that the global attractor belongs to some Gevrey class. This implies that its elements are real analytic functions in the spatial variables. A similar result is well-known for a class of semilinear parabolic equations (see, e.g.,[12], [14] and [25]). However, the method of proof for the parabolic case relies substantially on regularizing effects for parabolic equations (the solution at any time $t > 0$ is smoother than its initial value). This effect makes it possible to prove the existence of an absorbing set consisting of real analytic functions. This implies that every solution becomes a real analytic function in the spatial variable after some transient time. For the case considered here there is no regularizing effect. For this equation we can only prove the existence of a uniformly attracting set of real analytic functions. The main point here is to construct an appropriate decomposition of the evolution operator into decreasing and compact components. We rely on some ideas presented in [16] (see also the paper [24], where the analyticity of the attractor was established for a weakly damped and driven nonlinear Schrödinger equation and the paper [11] devoted to a similar problem for a class of dissipative nonlinear wave equations). We also note that the problem of Gevrey regularity of attractors and invariant sets was recently discussed in the abstract setting [18].

As an application of Theorem 1.3 on the analyticity of the attractor for problem (0.1)–(0.3), we prove that this problem in one spatial dimension possesses two determining nodes, i.e. the long-time behavior of the solutions is completely determined by their dynamics in two points inside the spatial interval $(0, L)$. We note that for the first time the relation between the Gevrey regularity and the existence of a small number of determining nodes was established in the paper [20] devoted to the Ginzburg–Landau equation (see also [12], [13] and the survey [9] for similar results for other equations). We also refer to [9] and [10] for a general discussion of the problem of existence of determining functionals for infinite-dimensional equations.

The paper is organized as follows. In Section 1 we introduce notation, give some background material on Gevrey classes and formulate our main results. Section 2 is devoted to the proof of the theorem on the Gevrey regularity of the attractor for the dynamical system generated by (0.1)–(0.3). In Section 3 we establish the existence of two determining nodes in one spatial dimension.

1 Preliminaries and statement of main results

Let
$$\mathcal{O} = (0, L) \times (0, L) \times \cdots \times (0, L) \subset \mathbb{R}^n.$$
We denote by $\dot{L}^2(\mathcal{O})$ the space of $L^2(\mathcal{O})$ functions with average zero, i.e.
$$\dot{L}^2(\mathcal{O}) = \left\{ u \in L^2(\mathcal{O}) : \int_{\mathcal{O}} u(x)\, dx = 0 \right\}.$$
For any $s \geq 0$ let us consider the Sobolev space
$$\dot{H}^s_{per}(\mathcal{O}) = \left\{ u \in H^s_{loc}(\mathbb{R}^n) : u(x + Le_j) = u(x),\ j = 1, \ldots, n,\ \int_{\mathcal{O}} u(x) dx = 0 \right\}.$$
Since $-\triangle$ generates a positive self-adjoint operator in the space $\dot{L}^2(\mathcal{O})$ with the domain $D(-\triangle) = \dot{H}^2_{per}(\mathcal{O})$, we can define the positive operator $A = (-\triangle)^{\frac{1}{2}}$

with the domain $D(A) = \dot{H}^1_{per}(\mathcal{O})$ and equip the space $\dot{H}^s_{per}(\mathcal{O})$ with the inner product

$$(u, v)_s = \int_{\mathcal{O}} ((-\triangle)^s u)(x) v(x)\, dx = (A^s u, A^s v), \quad s \geq 0,$$

where $(\cdot, \cdot)$ is the inner product in $\dot{L}^2(\mathcal{O})$. Below we will denote by $\| \cdot \|_s$ the corresponding norm in $\dot{H}^s_{per}(\mathcal{O})$ and by $\| \cdot \|$ the norm in $\dot{L}^2(\mathcal{O})$. We note that every element $u \in \dot{H}^s_{per}(\mathcal{O})$ can be represented in the form

$$u(x) = \sum_{j \in \mathbb{Z}^n} u_j \exp\left\{ i(j, x) \frac{2\pi}{L} \right\}, \tag{1.1}$$

where the Fourier coefficients u_j possess the properties $u_0 = 0$ and $\bar{u}_j = u_{-j}$ (the bar denotes the complex conjugate) and

$$\|u\|_s^2 = L^n \left(\frac{2\pi}{L} \right)^{2s} \sum_{j \in \mathbb{Z}^n} |j|^{2s} |u_j|^2 < \infty.$$

Let $s \geq 0$ and $\sigma > 0$. We introduce the Gevrey class $G^s_\sigma(\mathcal{O})$ as the Hilbert space consisting of real-valued functions of the form (1.1) such that

$$\|u\|^2_{G^s_\sigma(\mathcal{O})} = |u_0|^2 + L^n \left(\frac{2\pi}{L} \right)^{2s} \sum_{j \in \mathbb{Z}^n} |j|^{2s} \exp\left\{ \frac{4\sigma\pi}{L} |j| \right\} |u_j|^2 < \infty\,. \tag{1.2}$$

We denote the corresponding inner product by $(u, v)_{G^s_\sigma(\mathcal{O})}$. It is obvious that $G^s_0(\mathcal{O}) = H^s_{per}(\mathcal{O})$ for any $s \geq 0$ and $G^s_\sigma(\mathcal{O}) \subset H^m_{per}(\mathcal{O})$ for any $s \geq 0, \sigma > 0$ and $m \geq 0$. Moreover, every element $v \in G^s_\sigma(\mathcal{O})$, $\sigma > 0$ can be extended as an analytic function into the parallelepiped

$$\Pi^n = \left\{ z \in \mathbb{C}^n : \Re z \in \mathcal{O}, \Im z \in \left(-\frac{\sigma}{\sqrt{n}}, \frac{\sigma}{\sqrt{n}} \right)^n \right\}\,.$$

We denote by $\dot{G}^s_\sigma(\mathcal{O})$ the subspace of $G^s_\sigma(\mathcal{O})$ consisting of functions with average zero, i.e.

$$\dot{G}^s_\sigma(\mathcal{O}) = \left\{ u \in G^s_\sigma(\mathcal{O}) \ : \ \int_{\mathcal{O}} u(x)\, dx = 0 \right\} \equiv G^s_\sigma(\mathcal{O}) \cap \dot{L}^2(\mathcal{O}).$$

It is clear that $\dot{G}^s_\sigma(\mathcal{O}) = D(A^s e^{\sigma A})$, and

$$\|u\|^2_{G^s_\sigma(\mathcal{O})} = \|A^s e^{\sigma A} u\|^2 \quad \text{for all} \quad u \in \dot{G}^s_\sigma(\mathcal{O}), \tag{1.3}$$

where as above $A^2 = -\triangle$ with the periodic boundary conditions on $\mathcal{O}$ and $D(B)$ stands for the domain of the operator B.

Below we need the following Lemma (for the proof see [12]).

Lemma 1.1. *Let $s > \frac{n}{2}$, $\sigma \geq 0$ and let u and v be in the class $G^s_\sigma(\mathcal{O})$. Then $u \cdot v \in G^s_\sigma(\mathcal{O})$ and there exists a constant C_s, independent of σ, such that*

$$\|u \cdot v\|_{G^s_\sigma(\mathcal{O})} \leq C_s \|u\|_{G^s_\sigma(\mathcal{O})} \cdot \|v\|_{G^s_\sigma(\mathcal{O})}. \tag{1.4}$$

We consider a class of equations somewhat more general than in the hypotheses concerning the problem (0.1)-(0.3) given above. Our assumptions are as follows:

(A1) The problem (0.1)-(0.3) is well-posed in the class $C(0,\infty;\dot{H}^{s+1}_{per}(\mathcal{O}))$ for some $s > \frac{n}{2} - 1$, i.e. for any $u_0 \in \dot{H}^{s+1}_{per}(\mathcal{O})$ there exists a unique solution to problem (0.1)-(0.3) in the class $C(0,\infty;\dot{H}^{s+1}_{per}(\mathcal{O}))$ and this solution depends continuously on initial data.

(A2) The evolution operator S_t in the space $\dot{H}^{s+1}_{per}(\mathcal{O})$ generated by the formula $S_t u_0 = u(t)$, where $u(t) \in C(0,\infty;\dot{H}^{s+1}_{per}(\mathcal{O}))$ is the solution to problem (0.1)-(0.3), is dissipative in the space $\dot{H}^{s+1}_{per}(\mathcal{O})$, i.e. there exists $R_* > 0$ such that for every bounded set B from $\dot{H}^{s+1}_{per}(\mathcal{O})$ there exists $t_0(B)$ such that $\|S_t y\|_{s+1} \leq R_*$ for all $t \geq t_0(B)$ and $y \in B$.

(A3) $h(x)$ belongs to $\dot{G}^s_{\sigma_0}(\mathcal{O})$ for some $\sigma_0 > 0$ and for every $j = 1,\ldots,n$ the function $F_j(u)$ can be written in the form

$$F_j(u) = \sum_{k=1}^{\infty} b^j_k u^k,$$

where the coefficients $\{b^j_k\}$ satisfy

$$g_j(r) = \sum_{k=1}^{\infty} r^k |b^j_k| < \infty \quad \text{for all} \quad 0 < r < r_0, \quad j = 1,\ldots,m, \tag{1.5}$$

where r_0 is large enough.

The following proposition describes the cases when the assumptions (A1)-(A3) are valid.

Proposition 1.2. *Let $n \leq 3$, $h(x) \in \dot{G}^s_{\sigma_0}(\mathcal{O})$ and $F(u)$ has form (0.4). Then the properties (A1)-(A3) hold in the following cases: (a) $s = 0, n = 1$; (b) $s = 1, n \leq 2$; and (c) $s = 2, n \leq 3$.*

Proof. In the case (a) the assertion follows from [7, Theorem 5]. To obtain the proof for the other cases we use properties of the linearized problem (see, e.g., [7]) and the standard stepwise arguments. □

Our main result is the following assertion.

Theorem 1.3. *Assume that hypotheses (A1)–(A3) hold. Then the semigroup S_t generated in $\dot{H}^{s+1}_{per}(\mathcal{O})$ by equations (0.1)–(0.3) possesses a global attractor $\mathcal{A}$. This attractor $\mathcal{A}$ belongs to the class $\dot{G}^{s+1}_\sigma(\mathcal{O})$ for some $0 < \sigma \leq \sigma_0$ and*

$$\sup\left\{ \|u\|^2_{G^{s+1}_\sigma(\mathcal{O})} \ : \ u \in \mathcal{A} \right\} < \infty.$$

Thus all elements of the global attractor are real analytic functions in the spatial variables.

Theorem 1.3 also makes it possible to prove the following assertion on the existence of two determining nodes for problem (0.1)–(0.3) in the case of one spatial dimension.

Theorem 1.4. *Assume that $n = 1$ and hypotheses (A1)–(A3) hold with $s = 1$. Let x_1 and x_2 be two nodes such that $0 \le x_1 < x_2 \le L$. Let $u(t)$ and $u^*(t)$ be two solutions to problem (0.1)–(0.3) from the class $C(0,\infty;\dot H^2_{per}(\mathcal{O}))$. Then there exists Δ_0, independent of u and u^*, such that if $x_2 - x_1 \le \Delta_0$, then the condition*

$$\lim_{t\to\infty} |u(t,x_j) - u^*(t,x_j)| = 0, \quad j = 1, 2,$$

implies

$$\lim_{t\to\infty} \|u(t) - u^*(t)\|_2 = 0.$$

Theorem 1.4 means that the long-time behavior of any solution to problem (0.1)–(0.3) in the case of one spatial dimension case is completely determined by the values of the solution at any two nodes that are sufficiently close.

2 Gevrey regularity of the attractor

In this section we prove Theorem 1.3.

Let B be a bounded set in $\dot H^{s+1}_{per}(\mathcal{O})$. Assume that $u(t)$ is a solution to problem (0.1)-(0.3) with initial data $u_0 \in B$. Thus $u(t)$ satisfies the equation

$$u_t + aA^2u_t + bA^2u + \operatorname{div}\{F(u)\} = h, \qquad t \in \mathbb{R}^+. \tag{2.1}$$

Here $A^2 = -\triangle$ is the positive operator in $L^2(\mathcal{O})$ with the domain $D(A^2) = \dot H^2_{per}(\mathcal{O})$. Assumption $(A2)$ on the dissipativity implies that there exists $t_0 = t_0(B)$ such that

$$\sup\{\|u(t)\|_s^2 + a\|u(t)\|_{s+1}^2 \ : \ u_0 \in B\} \le R_0^2 \quad \text{for all} \quad t \ge t_0 = t_0(B), \tag{2.2}$$

where $R_0^2 = \left[a + (L/2\pi)^2\right] R_*^2$. Let

$$P_N(t) := P_N u(t),$$

where P_N is the orthogonal projector in $\dot L^2(\mathcal{O})$ onto the subspace

$$L_N = \left\{ u \in \dot L^2(\mathcal{O}) : u(x) = \sum_{0<|j|\le N} u_j \exp\left\{ i(j,x)\frac{2\pi}{L} \right\}, \bar u_j = u_{-j} \right\}. \tag{2.3}$$

In the subspace $Q_N\dot L^2(\mathcal{O})$ with $Q_N = I - P_N$ we consider the following auxiliary problem

$$w_t(t) + aA^2w_t(t) + bA^2w(t) + Q_N \operatorname{div}\{F(P_N(t) + w(t))\} = Q_N h \tag{2.4}$$

with the periodic boundary conditions:

$$w(t,x+L_ie_i)=w(t,x),\ w_{x_i}(t,x+L_ie_i)=w_{x_i}(t,x),\quad x\in\Gamma_i,\ t\in\mathbb{R}^+,\quad (2.5)$$

for $i=1,\dots,n$, and with the zero initial data at the time $t_0=t_0(B)$:

$$w(t_0,x)=0,\qquad x\in\mathcal{O}. \qquad (2.6)$$

We note that equation (2.4) formally arises as the projection of (2.1) onto the subspace $Q_N\dot{L}^2(\mathcal{O})$. However, the function $v(t)=P_N(t)+w(t)$ is not necessarily a solution to the original problem (0.1)-(0.3), because of the zero initial data for $w(t)$ at time t_0. In the following lemma we show that a solution $w(t)$ of the problem (2.4)-(2.6) belongs to the Gevrey class in the spatial variables and approximates the solution $u(t)$ to the original problem as $t\to\infty$. These properties allow us to prove the existence of a uniformly attracting set bounded in some Gevrey class, and to invoke standard results on attractors for asymptotically compact dynamical systems (see, e.g., [17] or [27]).

Lemma 2.1. *There exists $N_0\ge 1$ such that for any $N\ge N_0$ we can find $0<\sigma\equiv\sigma_N\le\sigma_0$ such that problem (2.4)-(2.6) has a unique solution in the class $C(t_0,\infty;Q_N\dot{G}^s_\sigma(\mathcal{O}))$. Here $Q_N=I-P_N$ and P_N is the orthoprojector onto the space L_N defined by (2.3). This solution possesses the property*

$$\|w(t)\|^2_{G^s_\sigma(\mathcal{O})}+a\|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})}\le R^2\qquad \textit{for all}\quad t\ge t_0=t_0(B),\quad N\ge N_0, \qquad (2.7)$$

with $R>0$ independent of N.

Proof. Instead of (2.4) we consider in the space $Q_{M,N}\dot{L}^2(\mathcal{O})$ the following equation

$$w_t(t)+aA^2w_t(t)+bA^2w(t)+Q_{M,N}\operatorname{div}\{F(P_N(t)+w(t))\}=Q_{M,N}h\ , \quad (2.8)$$

where $Q_{M,N}=P_M-P_N$ with $M>N$. It is obvious that this equation has a unique solution

$$w_M(t,x)=\sum_{N\le|j|\le M}w_j(t)\exp\left\{i(j,x)\frac{2\pi}{L}\right\},\qquad w_M(t_0,x)=0,$$

on some interval $(t_0,t_0+T_{M,N})$. If we prove the uniform estimate (2.7) for $w_M(t)$ on this existence interval, then we are able to conclude that this solution can be continued on the half-axis (t_0,∞). Then we will let $M\to\infty$ and obtain existence of a solution to the problem (2.4)-(2.6) with the property (2.7). It is easy to see that this solution is unique. Thus we need to prove only the uniform estimate (2.7) for $w_M(t)$ on the interval of existence. Below we omit the subscript M for the sake of notational simplicity.

We multiply equation (2.8) by $A^{2s}e^{2\sigma A}w$ with $\sigma\le\sigma_0$ in $L^2(\mathcal{O})$. Using (1.3), we get

$$\frac{d}{2dt}\left(\|w(t)\|^2_{G^s_\sigma(\mathcal{O})}+a\|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})}\right)+b\|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})}+$$
$$\left(Q_N\operatorname{div}\{F(P_N(t)+w(t))\},A^{2s}e^{2\sigma A}w(t)\right)=\left(Q_Nh,A^{2s}e^{2\sigma A}w(t)\right). \qquad (2.9)$$

Since $(Q_N h, A^{2s}e^{2\sigma A}w) = (A^s e^{\sigma A} Q_N h, A^s e^{\sigma A} w)$, it is easy to see that

$$|(Q_N h, A^{2s}e^{2\sigma A}w)| \le \frac{1}{4\epsilon_1}\|h\|^2_{G^s_\sigma(\mathcal{O})} + \epsilon_1 \|w(t)\|^2_{G^s_\sigma(\mathcal{O})}$$

for any $\epsilon_1 > 0$. Now we estimate the nonlinear term. Since

$$\begin{aligned} & (Q_N \operatorname{div}\{F(P_N(t)+w(t))\}, A^{2s}e^{2\sigma A}w(t)) \\ = & \sum_{i=1}^n (A^s e^{\sigma A} Q_N F_i(P_N(t)+w(t)), A^s e^{\sigma A} w_{x_i}(t)), \end{aligned}$$

we have

$$\begin{aligned} & |\big(Q_N \operatorname{div}\{F(P_N(t)+w(t))\}, A^{2s}e^{2\sigma A}w(t)\big)| \\ \le & \sum_{i=1}^n \|Q_N F_i(P_N(t)+w(t))\|_{G^s_\sigma(\mathcal{O})} \|w(t)\|_{G^{s+1}_\sigma(\mathcal{O})} \\ \le & \frac{n}{4\epsilon_2} \sum_{i=1}^n \|Q_N F_i(P_N(t)+w(t))\|^2_{G^s_\sigma(\mathcal{O})} + \epsilon_2 \|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})} \end{aligned}$$

for any positive ϵ_2. Therefore from (2.9) we get:

$$\begin{aligned} & \frac{d}{2dt}V(t) + (b-\epsilon_2)\|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})} - \epsilon_1 \|w(t)\|^2_{G^s_\sigma(\mathcal{O})} \\ \le & \frac{n}{4\epsilon_2} \sum_{i=1}^n \|Q_N F_i(P_N(t)+w(t))\|^2_{G^s_\sigma(\mathcal{O})} + \frac{1}{4\epsilon_1}\|h\|^2_{G^s_\sigma(\mathcal{O})}, \end{aligned} \tag{2.10}$$

where

$$V(t) = \|w(t)\|^2_{G^s_\sigma(\mathcal{O})} + a\|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})} .$$

Since

$$\|w\|^2_{G^s_\sigma(\mathcal{O})} \le \left(\frac{L}{2\pi(1+N)}\right)^{2(s^*-s)} \|w\|^2_{G^{s^*}_\sigma(\mathcal{O})}, \quad w \in Q_N G^{s^*}_\sigma(\mathcal{O}),\ s < s^*, \tag{2.11}$$

choosing ϵ_1 and ϵ_2 in an appropriate way from (2.10) we get constants $\mu > 0$, C_1 and C_2 independent of N such that

$$\frac{d}{dt}V(t) + 2\mu V(t) \le C_1 \sum_{i=1}^n \|Q_N F_i(P_N(t)+w(t))\|^2_{G^s_\sigma(\mathcal{O})} + C_2\|h\|^2_{G^s_\sigma(\mathcal{O})} . \tag{2.12}$$

Now we estimate the values $\|Q_N F_i(P_N(t)+w(t))\|^2_{G^s_\sigma(\mathcal{O})}$ relying on the representation of $F_i(P_N(t)+w(t))$ in the form

$$\begin{aligned} & F_i(P_N(t)+w(t)) = \\ = & F_i(P_N(t)) + F_i'(P_N(t))w(t) + \int_0^1 (1-\tau)F_i''(P_N(t)+\tau w(t))w(t)^2 \, d\tau . \end{aligned}$$

Let s^* be any number with the property $\max\{\frac{n}{2}, s\} < s^* < s+1$. Using Lemma 1.1 we have

$$\begin{aligned} & \|F_i(P_N(t))\|_{G_\sigma^{s^*}(\mathcal{O})} \le \sum_{k=1}^{\infty} |b_k^i| \cdot \|(P_N(t))^k\|_{G_\sigma^{s^*}(\mathcal{O})} \\ \le & \sum_{k=1}^{\infty} |b_k^i| C_{s^*}^{k-1} \|P_N(t)\|^k_{G_\sigma^{s^*}(\mathcal{O})} \le C_{s^*} g_i\left(C_{s^*}\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})}\right). \end{aligned}$$

Here C_s is the constant from Lemma 1.1 and $g_i(r)$ is defined by (1.5). In a similar way we obtain

$$\begin{aligned} & \|F_i'(P_N(t))w(t)\|_{G_\sigma^{s^*}(\mathcal{O})} \le C_{s^*}\|F_i'(P_N(t))\|_{G_\sigma^{s^*}(\mathcal{O})}\|w(t)\|_{G_\sigma^{s^*}(\mathcal{O})} \\ \le & \ C \cdot g_i'(C_{s^*}\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})})\|w(t)\|_{G_\sigma^{s^*}(\mathcal{O})}, \end{aligned}$$

where $g_i'(r)$ is the derivative of $g_i(r)$ and the constant C does not depend on N. The same argument gives

$$\begin{aligned} & \|(F_i''(P_N(t)+\tau w(t))w(t), w(t))\|_{G_\sigma^{s^*}(\mathcal{O})} \le \\ \le & \ C_{s^*}^2 \|F_i''(P_N(t)+\tau w(t))\|_{G_\sigma^{s^*}(\mathcal{O})}\|w(t)\|^2_{G_\sigma^{s^*}(\mathcal{O})} \\ \le & \ C \cdot g_i''(C_{s^*}\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})} + C_{s^*}\|w(t)\|_{G_\sigma^{s^*}(\mathcal{O})}) \cdot \|w(t)\|^2_{G_\sigma^{s^*}(\mathcal{O})}, \end{aligned}$$

where g_i'' is the second derivative of $g_i(r)$ and the constant C is independent of N. Thus we have

$$\begin{aligned} \mathcal{F}(N, s^*) \ \equiv \ & \sum_{i=1}^{n} \|Q_N F_i(P_N(t)+w(t))\|^2_{G_\sigma^{s^*}(\mathcal{O})} \\ \le \ & G_0(C_{s^*}\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})}) + G_1(C_{s^*}\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})}) \cdot \|w(t)\|^2_{G_\sigma^{s^*}(\mathcal{O})} \\ & + G_2(C_{s^*}\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})} + C_{s^*}\|w(t)\|_{G_\sigma^{s^*}(\mathcal{O})}) \cdot \|w(t)\|^4_{G_\sigma^{s^*}(\mathcal{O})}. \end{aligned}$$

Here $G_k(r) = C \cdot \sum_{i=1}^n [g_i^{(k)}(r)]^2$, $k \in \{0,1,2\}$, where $g_i^{(k)}(r)$ is the derivative of $g_i(r)$ of order k and C does not depend on N.

Since $s^* < s+1$, from (2.2) we have

$$\|P_N(t)\|_{G_\sigma^{s^*}(\mathcal{O})} \le c_0 e^{\tilde{\sigma}N}\|P_N(t)\|_{s+1} \le c_0 e^{\tilde{\sigma}N} R_0, \qquad t \ge t_0 = t_0(B),$$

where $\tilde{\sigma} = 2\pi\sigma/L$ and c_0 is independent of N. Therefore if we choose $\sigma = N^{-1}$, we obtain

$$\mathcal{F}(N, s^*) \le C_0 + C_1\|w(t)\|^2_{G_\sigma^{s^*}(\mathcal{O})} + G_2(C_2 + C_{s^*}\|w(t)\|_{G_\sigma^{s^*}(\mathcal{O})}) \cdot \|w(t)\|^4_{G_\sigma^{s^*}(\mathcal{O})}$$

for all $t \ge t_0$, where C_j does not depend on N, $j \in \{0,1,2\}$. Consequently (2.11) implies

$$\begin{aligned} & \sum_{i=1}^{n} \|Q_N F_i(P_N(t)+w(t))\|^2_{G_\sigma^{s}(\mathcal{O})} \le \left(\frac{L}{2\pi}\right)^{s^*-s} \mathcal{F}(N, s^*) \\ \le & \ C_0 + C_1 N^{-2\delta}\|w(t)\|^2_{G_\sigma^{s+1}(\mathcal{O})} \\ & + N^{-4\delta} G_2(C_2 + C_{s^*}\|w(t)\|_{G_\sigma^{s+1}(\mathcal{O})})\|w(t)\|^4_{G_\sigma^{s+1}(\mathcal{O})}, \end{aligned}$$

where $\delta = s + 1 - s^* > 0$. Therefore from (2.12) we get

$$\frac{d}{dt}V + \mu V \leq C_1 + C_2 N^{-4\delta} V^2 G_2(C_3 + C_4 V^{\frac{1}{2}}) \tag{2.13}$$

for all N large enough, where the constants $C_j > 0$ do not depend on N. It is clear that for N large enough the function

$$F(V) = -\mu V + C_1 + C_2 N^{-4\delta} V^2 \cdot G_2(C_3 + C_4 \cdot V^{\frac{1}{2}})$$

has a simple root $V_0 = V_0(N) > 0$ such that $F(V) > 0$ for $V \in (0, V_0)$. Since $V(t_0) = 0$, this property of $F(V)$ and inequality (2.13) imply that $V(t) \leq V_0$ for all $t \geq t_0$. Moreover, since $V_0(N) \to C_1 \mu^{-1}$ for $N \to \infty$, we have that $V_0(N) \leq V_0^*$, where V_0^* does not depend on N. Therefore, we have

$$\|w(t)\|^2_{G^s_\sigma(\mathcal{O})} + a\|w(t)\|^2_{G^{s+1}_\sigma(\mathcal{O})} \leq V_0^* = R^2, \quad t \geq t_0, \tag{2.14}$$

for N large enough with any $0 \leq \sigma \leq N^{-1} \leq \sigma_0$. Thus we obtain (2.7). $\square$

Lemma 2.2. *Let $u(t)$ be a solution to the problem (2.1) and (0.2) with initial data $y = u_0 \in B$, where B is a bounded set in $\dot{H}^{s+1}_{per}(\mathcal{O})$, such that (2.2) holds. Assume that $w(t)$ is the solution to problem (2.4)-(2.6). Then there exists $N_0 >$ 0 such that we have*

$$\lim_{t\to\infty} \sup_{y \in B} \{\|Q_N u(t) - w(t)\|^2_s + a\|Q_N u(t) - w(t)\|^2_{s+1}\} = 0 \tag{2.15}$$

for every $N \geq N_0$, where $Q_N = I - P_N$.

Proof. Let $v(t) = Q_N u(t) - w(t)$. Then $v(t)$ is a solution to the problem

$$\begin{aligned} & v_t(t) + aAu_t(t) + bAu(t) + \\ & + Q_N \operatorname{div}\{F(u(t)) - F(P_N(t) + w(t))\} = 0, \end{aligned} \tag{2.16}$$

with periodic boundary conditions and with the initial data:

$$v(t_0) = Q_N u(t_0). \tag{2.17}$$

We multiply equation (2.16) by $A^{2s}v$ in $\dot{L}^2(\mathcal{O})$ and get

$$\begin{aligned} & \frac{d}{2dt}\left(\|v(t)\|^2_s + a\|v(t)\|^2_{s+1}\right) + b\|v(t)\|^2_{s+1} \\ \leq\ & |(Q_N \operatorname{div}\{F(u(t)) - F(P_N(t) + w(t))\}, A^{2s}v(t))| \equiv \mathcal{D}(N,t). \end{aligned} \tag{2.18}$$

Now using integration by parts and the Hölder inequality for the right hand side of inequality (2.18) we get

$$\begin{aligned} \mathcal{D}(N,t) & \leq \sum_{i=1}^n |(A^s Q_N (F_i(u(t)) - F_i(P_N(t) + w(t)), A^s v_{x_i}(t)))| \\ & \leq \sum_{i=1}^n \|Q_N(F_i(u(t)) - F_i(P_N(t) + w(t)))\|_s \|v(t)\|_{s+1} \\ & \leq \epsilon \|v(t)\|^2_{s+1} + \frac{n}{4\epsilon} \sum_{i=1}^n \|Q_N \widetilde{F}_i(u(t), w(t))\|^2_s \ , \end{aligned}$$

where $\epsilon > 0$ is arbitrary and

$$\widetilde{F}_i(u(t), w(t)) = F_i(u(t)) - F_i(P_N(t) + w(t)), \quad \text{for } i = 1, 2, \dots, n.$$

Let

$$W(t) = \|v(t)\|_s^2 + a\|v(t)\|_{s+1}^2.$$

As above it is easy to prove that there exist positive constants μ and C independent of N such that

$$\frac{d}{dt}W(t) + 2\mu W(t) \leq C \sum_{i=1}^{n} \|Q_N \widetilde{F}_i(u(t), w(t))\|_s^2 \ . \tag{2.19}$$

Since $P_N(t) + w(t) = u(t) - v(t)$, we have

$$\widetilde{F}_i(u, w) = \int_0^1 F_i'(u - \tau v)v \ d\tau \ .$$

Therefore

$$\|Q_N \widetilde{F}_i(u, w)\|_s \leq c_0 \|Q_N \widetilde{F}_i(u, w)\|_{s^*} \leq c_0 \int_0^1 \|F_i^{'}(u - \tau v)v\|_{s^*} \ d\tau,$$

where s^* is any number with the property $\max\{\frac{n}{2}, s\} < s^* < s + 1$ and c_0 is independent of N. Lemma 1.1 with $\sigma = 0$ and assumption (A3) imply

$$\|F_i^{'}(u - \tau v)v\|_{s^*} \leq C \cdot \|F_i^{'}(u - \tau v)\|_{s^*} \|v\|_{s^*} \leq C_1 \cdot g_i'(C_2 \|u - \tau v\|_{s^*}) \|v\|_{s^*} \ .$$

Hence

$$\sum_{i=1}^{n} \|Q_N \widetilde{F}_i(u(t), w(t))\|_s^2 \leq C \cdot \sum_{i=1}^{n} [g_i'(C_2 \|u(t) - \tau v(t)\|_{s^*})]^2 \|v(t)\|_{s^*}^2 \ .$$

¿From (2.2) and (2.14) with $\sigma = 0$ we have

$$\|u(t) - \tau v(t)\|_{s^*}^2 \leq c_0 \|u(t) - \tau v(t)\|_{s+1}^2 \leq C(R_0, R) \quad \text{for all } t \geq t_0 = t_0(B).$$

Consequently using (2.11) with $\sigma = 0$ we obtain

$$\sum_{i=1}^{n} \|Q_N \widetilde{F}_i(u(t), w(t))\|_s^2 \leq C(R_0, R) \cdot (1 + N)^{-2(s+1-s^*)} \|v(t)\|_{s+1}^2$$

for all $t \geq t_0 = t_0(B)$. Thus (2.19) implies that there exists N_0 such that

$$\frac{d}{dt}W(t) + \mu W(t) \leq 0$$

for all $t \geq t_0 = t_0(B)$, $N \geq N_0$. This inequality implies $W(t) \leq e^{-\mu(t-t_0)}W(t_0)$ for all $t \geq t_0 = t_0(B)$ and $N \geq N_0$. Therefore we obtain (2.15). This completes the proof of Lemma 2.2. $\square$

Lemmas 2.1 and 2.2 allow us now to conclude the proof of Theorem 1.3.

Proof of Theorem 1.3. Let us fix N such that the assertions of Lemmas 2.1 and 2.2 are valid. Assume that B is a bounded set in $\dot{H}^{s+1}_{per}(\mathcal{O})$ and $u(t) = S_t u_0$, where $u_0 \in B$. Let $w(t)$ be the corresponding solution to problem (2.4)–(2.6). ¿From relation (2.2) and Lemma 2.1 we have

$$\|P_N u(t) + w(t)\|^2_{G^s_\sigma(\Omega)} + a\|P_N u(t) + w(t)\|^2_{G^{s+1}_\sigma(\Omega)} \le \bar{R}^2 \equiv c(R_0^2 + R^2)$$

for all $t \ge t_0(B)$. Since $S_t u_0 = (Q_N u(t) - w(t)) + (P_N u(t) + w(t))$, Lemma 2.2 implies that the set

$$\mathcal{G} = \left\{ u \in \dot{G}^{s+1}_\sigma(\mathcal{O}) \ : \ \|u\|^2_{G^s_\sigma(\Omega)} + a\|u\|^2_{G^{s+1}_\sigma(\Omega)} \le \bar{R}^2 \right\}$$

is uniformly attracting for S_t. Since $\mathcal{G}$ is compact in $\dot{H}^{s+1}_{per}(\mathcal{O})$, the standard theorems on the existence of attractors for the asymptotically compact case (see, e.g., [4], [17] or [27]) imply that $\mathcal{A} \subset \mathcal{G}$. $\square$

3 Determining nodes ($n = 1$)

In this section we prove Theorem 1.4 on the existence of determining nodes for problem (0.1)–(0.3) in one spatial dimension.

Lemma 3.1. *Assume that $n = 1$ and that assumptions (A1)-(A3) are valid with $s = 1$. Let $u_0 \in \dot{H}^2_{per}(0, L)$. Then the solution $u(t, x)$ to the problem (0.1)-(0.3) belongs to the space $C^1([0, +\infty); \dot{H}^2_{per}(0, L))$ and there exists $R > 0$ and $t_0 = t_0(u_0)$ such that*

$$\|u_t(t)\|_2^2 + \|u(t)\|_2^2 \le R^2, \qquad t \ge t_0 \ . \tag{3.1}$$

Proof. By (A1) $u \in C([0, +\infty); \dot{H}^2_{per}(0, L))$. From hypothesis (A2) we obtain that $\|u(t)\|_2 \le R_*$ for $t \ge t_0$ with some t_0. Therefore (0.1) implies that

$$(1 - a\partial_x^2)u_t \in C([0, +\infty); \dot{L}^2(0, L))$$

and

$$\|(1 - a\partial_x^2)u_t(t)\| \le C(R_*) \quad \text{for} \quad t \ge t_0.$$

Thus $u_t \in C^1([0, +\infty); \dot{H}^2_{per}(0, L))$ and $\|u_t(t)\|_2 \le C(R_*)$ for $t \ge t_0$. $\square$

Lemma 3.2. *Assume that x_1 and x_2 are two nodes such that $0 \le x_1 < x_2 \le L$. Let $u^1(t, x)$ and $u^2(t, x)$ be two solutions to problem (0.1)-(0.3) for $n = 1$ from the class $C^1(0, +\infty; \dot{H}^2_{per}(0, L))$ such that*

$$\|u^k_t(t)\|_2^2 + \|u^k(t)\|_2^2 \le R^2 \quad \textit{for} \quad t \ge t_0,\ k = 1, 2, \tag{3.2}$$

for some $R > 0$ and $t_0 \geq 0$. Then there exists $\Delta_0 = \Delta_0(R) > 0$ and $\mu > 0$ such that under the condition $0 < x_2 - x_1 < \Delta_0$, we have

$$\|u(t)\|_\Delta^2 + a\|u_x(t)\|_\Delta^2 \leq C_1 e^{-\mu(t-s)} + C_2 \int_s^t e^{-\mu(t-s)} \max_{l=1,2} |u(\tau, x_l)| \, d\tau \tag{3.3}$$

for any $t \geq s \geq t_0$, where $u(t) = u^1(t) - u^2(t)$ and C_j are positive constants depending on R. Here and below we use the notation: $\Delta = (x_1, x_2)$, $|\Delta| = x_2 - x_1$ and

$$\|u\|_\Delta^2 = \int_{x_1}^{x_2} u^2(x)\, dx, \quad \|u\|_{1,\Delta}^2 = \|u\|_\Delta^2 + a\|u_x\|_\Delta^2.$$

Proof. We use the same approach as in [20]. From (0.1) we have that $u(t, x)$ is a solution to the equation

$$u_t - au_{txx} - bu_{xx} + \big(F(u^1) - F(u^2)\big)_x = 0, \quad x \in (0, L),\ t > 0.$$

Multiplying this equation by $u(t, x)$ and integrating by parts from x_1 to x_2 we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u(t)\|_{1,\Delta}^2 + b\|u_x(t)\|_\Delta^2 \qquad (3.4)\\ = \quad & \eta(t, x_2) - \eta(t, x_1) + \delta(t, x_2) - \delta(t, x_1) - \int_{x_1}^{x_2} \big(F(u^1) - F(u^2)\big)_x u\, dx\,, \end{aligned}$$

where $\eta(t, x) = au(t, x)u_{xt}(t, x)$ and $\delta(t, x) = bu(t, x)u_x(t, x)$. Since $\dot{H}^1_{per}(0, L)$ is embedded into $C([0, L])$, it is easy to see from (3.2) that

$$|\eta(t, x_2) - \eta(t, x_1)| + |\delta(t, x_2) - \delta(t, x_1)| \leq C_R \max_{l=1,2} |u(t, x_l)|$$

and

$$\Big| \int_{x_1}^{x_2} \big(F(u^1) - F(u^2)\big)_x u\, dx \Big| \leq C_{R,\epsilon}\|u\|_\Delta^2 + \epsilon\|u_x\|_\Delta^2,$$

where ϵ is any positive number. Choosing $\epsilon = b/2$ and substituting all the estimates into (3.4) we get the following inequality:

$$\frac{d}{dt}\|u(t)\|_{1,\Delta}^2 + \frac{b}{a}\|u(t)\|_{1,\Delta}^2 \leq C_R^1 \max_{l=1,2} |u(t, x_l)| + C_R^2\|u(t)\|_\Delta^2. \tag{3.5}$$

Since

$$u(x)^2 - u(x_1)^2 = 2\int_{x_1}^{x} u_x(\xi) \cdot u(\xi)\, d\xi \leq 2\|u\|_\Delta \cdot \|u_x\|_\Delta\ ,$$

it is easy to see that

$$\|u(t)\|_\Delta^2 \leq 2|\Delta||u(t, x_1)|^2 + 4|\Delta|^2\|u_x(t)\|_\Delta^2\ .$$

Substituting this into (3.5) we get

$$\frac{d}{dt}\|u(t)\|_{1,\Delta}^2 + \frac{b}{2a}\|u(t)\|_{1,\Delta}^2 \leq C_R \max_{l=1,2} |u(t, x_l)|$$

for $|\Delta|$ small enough. This inequality implies (3.3). $\square$

Now we have everything at hand to prove Theorem 1.4.

Proof of Theorem 1.4. Assume that

$$\lim_{t\to\infty} |u(t,x_j) - u^*(t,x_j)| = 0, \quad j = 1,2,$$

for $0 < x_2 - x_1 < \Delta_0$ with Δ_0 from Lemma 3.2. Then Lemma 3.2 implies that

$$\lim_{t\to\infty} \{\|u(t) - u^*(t)\|_\Delta + a\|u_x(t) - u_x^*(t)\|_\Delta\} = 0. \tag{3.6}$$

Assume $\lim_{t\to\infty} \|u(t) - u^*(t)\|_2 = 0$ does not hold. In this case, since the global attractor $\mathcal{A}$ is a compact set in $\dot{H}^2_{per}(0,L)$ and $\mathrm{dist}_{H^2_{per}}(u(t),\mathcal{A}) \to 0, \quad t \to +\infty$, for any solution $u(t)$, there exists a sequence $\{t_n\}$ and elements $y(x)$ and $y^*(x)$ from $\mathcal{A}$ such that

$$\lim_{n\to\infty} \|u(t_n) - u^*(t_n)\|_2 > 0 \tag{3.7}$$

and $\lim_{n\to\infty} \|u(t_n) - y\|_2 = 0$ as well as $\lim_{n\to\infty} \|u^*(t_n) - y^*\|_2 = 0$. It follows from (3.6) that $y(x) = y^*(x)$ for $x \in \Delta = (x_1, x_2)$. Since by Theorem 1.3 the elements of the attractor $\mathcal{A}$ are real analytic functions in the spatial variable, we have that $y(x) = y^*(x)$ for any $x \in (0,L)$. This contradicts (3.7) and completes the proof. $\square$

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